On quasi-separative semigroups

Yu. I. Krasilnikova, B. V. Novikov

ABSTRACT. We built some congruences on semigroups, from where a decomposition of quasi-separative semigroups was obtained.

1 Introduction

The research of separative semigroups was being begun from the famous paper of Hewitt and Zuckerman [3, 5], where, in particular, they proved that any commutative separative semigroup is isomorphic to a semilattice of cancellative semigroups. An generalization for the noncommutative case has been made by Burmistrovich [1] and independently by Petrich [4]. Drazin [2] introduced the term 'quasi-separativity' and studied connections between it and others semigroup properties (inversity, regularity etc).

We shall follow the terminology proposed by Drazin:

Definition 1 *A semigroup S is called separative*[1] *if*

$$\begin{cases} x^2 = xy \\ y^2 = yx \end{cases} \Longrightarrow\ x = y \qquad \text{and} \qquad \begin{cases} x^2 = yx \\ y^2 = xy \end{cases} \Longrightarrow\ x = y \tag{1}$$

for all $x, y \in S$.

A semigroup S is called quasi-separative if

$$x^2 = xy = yx = y^2 \Longrightarrow x = y \tag{2}$$

for all $x, y \in S$.

Drazin also showed that in the definition of quasi-separativity we can replace (2) by the next condition

$$x^2 = xy = y^2 \Longrightarrow x = y. \tag{3}$$

It often simplifies considerably proofs of assertions.

[1] Burmistrovich [1] called it *weakly cancellative*

The main result of this paragraph is an extension of the Burmistrovich's theorem (Theorem 3): any quasi-separative semigroup is decomposable into a semilattice of subsemigroups, which are called *quasi-cancellative* by us. With this aim we previously build certain congruences on arbitrary semigroup (Theorem 1); they give semilattice decompositions in the quasi-separative case. As a corrolary, in Sect. 4 we consider an intermediate class of semigroups (*weakly balanced semigroups*) between separative and quasi-separative ones and discuss the connections between them.

2 Relation Ω

Let S be an arbitrary semigroup. By analogy with [2], we define two binary relations $E(a), F(a) \subset S \times S$ for every element $a \in S$:

$$E(a) = \{(x, y) \mid ax = ay\}, \quad F(a) = \{(x, y) \mid xa = ya\}.$$

The next properties of these relations are obvious:

$$E(b) \subset E(ab) \tag{4}$$

$$F(a) \subset F(ab) \tag{5}$$

$$bE(ab) \subset E(a) \tag{6}$$

$$F(ab)a \subset F(b) \tag{7}$$

(here and below for a binary relation $R \subset S \times S$ and for an element $x \in S$ the relation $\{(xa, xb) \mid (a, b) \in R\}$ is denoted by xR; analogously, Rx).

In what follows, the main tools for our studying will be the relations $\Omega \subset S \times S$, which satisfy the next conditions:

$$\forall a \quad \Omega \cap E(a) = \Omega \cap F(a) \tag{8}$$

$$\forall a, b \quad b(\Omega \cap E(ab)) \subset \Omega \tag{9}$$

$$\forall a, b \quad (\Omega \cap F(ab))a \subset \Omega \tag{10}$$

and the equivalences $\sim_\Omega$ on S corresponding to these relations:

$$a \sim_\Omega b \Longleftrightarrow \Omega \cap E(a) = \Omega \cap E(b).$$

According to (8), such definition is equal to the following:

$$a \sim_\Omega b \Longleftrightarrow \Omega \cap F(a) = \Omega \cap F(b).$$

Lemma 1 *For all elements $a, b \in S$ and a relation Ω which satisfies the conditions (8)-(10)*

$$\Omega \cap E(a) \subset \Omega \cap E(ab) \cap E(ba).$$

Proof. By (8) we have:

$$\Omega \cap E(a) = (\Omega \cap E(a)) \cap (\Omega \cap F(a)).$$

From here, using (4) and (5), we get:

$$\Omega \cap E(a) \subset (\Omega \cap E(ba)) \cap (\Omega \cap F(ab)) = \Omega \cap E(ba) \cap E(ab)$$

(the last equality follows from (8)). ■

Our first result is fulfilled for an arbitrary semigroup:

Theorem 1 *Let $\Omega \subset S \times S$ satisfies the conditions (8)-(10). Then the equivalence $\sim_\Omega$ is a congruence on S.*

Proof. Let $a, b, c \in S$ and $a \sim_\Omega b$. Obviously, for the proving the right compatibility of $\sim_\Omega$ it is enough to verify the inclusion

$$\Omega \cap E(ac) \subset \Omega \cap E(bc).$$

Let $(x, y) \in \Omega \cap E(ac)$. Owing to (9) $(cx, cy) \in \Omega$. On the other hand, (6) implies the inclusion $(cx, cy) \in cE(ac) \subset E(a)$. Therefore,

$$(cx, cy) \in \Omega \cap E(a) = \Omega \cap E(b).$$

From here it follows that $(x, y) \in \Omega \cap E(bc)$.

Similarly, by (7) and (10) the left compatibility can be proved. ■

Example. Let S be a commutative semigroup, $\Omega = S \times S$. Then the conditions of Theorem 1 are true and the equivalence

$$a \sim b \Longleftrightarrow E(a) = E(b)$$

is a congruence relation.

3 A decomposition of quasi-separative semigroups

In this section we apply the preceding theorem to quasi-separative semigroups.

Note that the definition of quasi-separativity in the form (3) may be formulated in terms of the relations $E(a)$ and $F(a)$:

$$(a, b) \in E(a) \cap F(b) \Longrightarrow a = b \tag{11}$$

for all $a, b \in S$.

Theorem 2 *Let Ω be a relation on quasi-separative semigroup S which satisfies the conditions (8)-(10). Then $S/{\sim_\Omega}$ is a semilattice.*

Proof. First, show that $S/{\sim_\Omega}$ is a band. In order to verify this statement it is sufficient to justify that the equality $\Omega \cap E(a) = \Omega \cap E(a^2)$ is right for any $a \in S$.

An inclusion

$$\Omega \cap E(a) \subset \Omega \cap E(a^2)$$

at once follows from Lemma 1. Conversely, if $(x, y) \in \Omega \cap E(a^2)$, then $(ax, ay) \in E(a)$. Moreover, owing to (9)

$$(ax, ay) \in a(\Omega \cap E(a^2)) \subset \Omega,$$

hence, $(ax, ay) \in \Omega \cap E(a)$. By Lemma 1

$$(ax, ay) \in \Omega \cap E(ax) \cap E(xa) \cap E(ya) \cap E(ay),$$

whence, in particular,

$$(ax, ay) \in \Omega \cap E(ax) \cap E(ay) = \Omega \cap E(ax) \cap F(ay)$$

by the condition (8). From (11) we obtain $ax = ay$, that is $(x, y) \in E(a)$. Therefore, $\Omega \cap E(a^2) \subset \Omega \cap E(a)$ and the first part of Theorem is proved.

Now we shall prove that $S/{\sim_\Omega}$ is commutative, viz. that $\Omega \cap E(ab) = \Omega \cap E(ba)$. The successive using the properties (4), (8) and (5) gives us:

$$\Omega \cap E(ab) \subset \Omega \cap E(bab) = \Omega \cap F(bab) \subset \Omega \cap F((ba)^2) = \Omega \cap E((ba)^2).$$

Since, as proved above, $S/{\sim}$ is a band, then

$$\Omega \cap E(ab) \subset \Omega \cap E(ba).$$

Analogously,

$$\Omega \cap E(ba) \subset \Omega \cap E(ab),$$

what completes the proof of Theorem. ■

Next assertion gives us a preliminary information about $\sim_\Omega$-classes. Denote by Δ_T the diagonal of Cartesian square $T \times T$.

Proposition 1 *If S is a quasi-separative semigroup, then each $\sim_\Omega$-class T satisfies the next condition for all $a \in T$:*

$$\Omega \cap E(a) \cap (T \times T) \subset \Delta_T.$$

Proof. Indeed, let $(x, y) \in \Omega \cap E(a) \cap (T \times T)$. Since $x \sim_\Omega y \sim_\Omega a$, then

$$(x, y) \in \Omega \cap E(a) = \Omega \cap E(x) \cap F(y),$$

whence, by (11), we have $x = y$. ■

Definition 2 *We call a semigroup S quasi-cancellative if the condition*

$$\begin{cases} \forall x, y \in S^1 \quad xby = xcy \Longleftrightarrow yxb = yxc \Longleftrightarrow byx = cyx \\ ab = ac. \end{cases}$$

implies $b = c$.

Obviously, every right- or left-cancellative semigroup is quasi-cancellative.

Our main result on structure of quasi-separative semigroups is the next

Theorem 3 *A semigroup is quasi-separative if and only if it is a semilattice of quasi-separative quasi-cancellative semigroups.*

Proof. *Necessity.* Denote a binary relation Ω_S on S:

$$\Omega_S = \{(x, y) \mid \forall a, b \in S^1 \quad axb = ayb \Longleftrightarrow xba = yba \Longleftrightarrow bax = bay\} \quad (12)$$

and verify the conditions (8)-(10) for it. Since for $a = 1$ we have:

$$xb = yb \Longleftrightarrow bx = by,$$

for any pair $(x, y) \in \Omega_S$, then obviously, (8) holds. Now we prove that Ω_S is left compatibility, from where (9) will follow.

Let $(x, y) \in \Omega_S$, $b \in S$. To prove that $(bx, by) \in \Omega_S$ one needs to check the fulfilment of the implications:

$$\forall c, d \in S^1 \quad cbxd = cbyd \Longleftrightarrow dcbx = dcby \Longleftrightarrow bxdc = bydc.$$

The implication $cbxd = cbyd \Longleftrightarrow dcbx = dcby$ immediately follows from the definition of Ω_S. Let $dcbx = dcby$. From (12) we obtain:

$$dcbx = dcby \Longrightarrow xdcb = ydcb.$$

Therefore

$$(bxdc)^2 = bx(dcbx)dc = b(xdcb)ydc = (bydc)^2$$

and quasi-separativity implies $bxdc = bydc$.

Similarly, if $bxdc = bydc$, then

$$bxdc = bydc \Longrightarrow dcbx = dcby \Longrightarrow xdcb = ydcb.$$

Hence

$$(cbxd)^2 = cbx(dcbx)d = c(bxdc)byd = (cbyd)^2$$

and $cbxd = cbyd$.

In the same way right compatibility is checked, and so the condition (10) is fulfilled.

Thus, $S/{\sim_{\Omega_S}}$ is a commutative band by Theorem 2. It remains to show that its components are quasi-cancellative.

Let suppose that the conditions of Definition 2 hold for some elements a, b, c, d from the $\sim_{\Omega_S}$-class $T \subset S$. It means that

$$(c, d) \in \Omega_T \cap E(a) \subset \Omega_S \cap E(a) = \Omega_S \cap E(b).$$

Hence $bc = bd$. Moreover, (8) implies $cb = db$. In particular, replacing b in the obtained equations by c and d, we get:

$$c^2 = cd = dc = d^2,$$

whence $c = d$.

Sufficiency. It is easy to see that any semilattice of quasi-separative semigroups is also quasi-separative. ■

4 Corollaries and Examples

In this section we show that Theorem 3 implies the theorem of Burmistrovich on the separative semigroups and obtain an assertion about certain intermediate class of semigroups.

Proposition 2 *Every separative quasi-cancellative semigroup is cancellative.*

Proof. Let S be separative and quasi-cancellative, $a, b, c \in S$, $ab = ac$. By Lemma 1 from [1] for all $x, y \in S$

$$xby = xcy \Longrightarrow byx = cyx \Longrightarrow yxb = yxc \Longrightarrow xby = xcy.$$

So, by quasi-separativity $b = c$.

To prove the right cancellativity we ought to apply the Lemma 1 [1] to the equality $ba = ca$ and to refer to the previous argumentation. ■

Corollary 1 (Burmistrovich's Theorem [1]) *A semigroup is separative if and only if it is isomorphic to a semilattice of cancellative semigroups.*■

Definition 3 *A semigroup S is called weakly cancellative [4] if for every $a, b, x, y \in S$*

$$\left\{ \begin{array}{l} ax = ay \\ xb = yb \end{array} \right. \tag{13}$$

implies $x = y$.

We call a semigroup S weakly balanced, if (13) implies

$$\left\{ \begin{array}{l} xa = ya \\ bx = by. \end{array} \right.$$

Obviously, every weakly cancellative semigroup is quasi-separative; but in general this is not hold in the weakly balanced case (for example, all commutative semigroups are weakly balanced). On the other hand, by abovementioned Lemma 1 [1] all separative semigroups are weakly balanced, so two next facts give a partial extension of Burmistrovich's theorem to the more wide class of semigroups.

Proposition 3 *If S is a quasi-cancellative weakly balanced semigroup, then S is also weakly cancellative.*

Proof. Let S be quasi-cancellative and weakly balanced, $a, b, x, y \in S$ and

$$ax = ay, \qquad xb = yb.$$

If $uxv = uyv$ for some elements $u, v \in S^1$, then by the weakly balancity from this last equality and from $axv = ayv$ we obtain $xvu = yvu$. Similarly, implications

$$xvu = yvu \Longrightarrow vux = vuy \Longrightarrow uxv = uyv.$$

can be obtained. Now $x = y$ because of quasi-cancellativity. ■

Corollary 2 *Every quasi-separative weakly balanced semigroup is isomorphic to a semilattice of weakly cancellative semigroups.* ■

We don't know whether the converse to the Corollary 2 is true. One can affirm only that a semilattice of weakly cancellative semigroups (which, evidently, is quasi-separative) satisfies the next condition:

$$\left\{ \begin{array}{lcl} a^2x & = & a^2y \\ xa^2 & = & ya^2 \end{array} \right. \Longrightarrow \left\{ \begin{array}{lcl} ax & = & ay \\ xa & = & ya \end{array} \right. \tag{14}$$

Really, it follows out of $a^2x = a^2y$ that ax, ay, xa, ya contain in the same component of the semilattice. From the antecedent of (14) we have:

$$\left\{ \begin{array}{lcl} (xa)(ax) & = & (xa)(ay) \\ (ax)(a^2x) & = & (ay)(a^2x) \end{array} \right.$$

Now weakly cancellativity implies $ax = ay$ and, similarly, $xa = ya$.

In conclusion we discuss the connections between considered classes of semigroups. They may be presented by a diagram:

$$\begin{tikzcd}
\boxed{\text{Separativity}} \arrow[r, Rightarrow] & \boxed{\begin{array}{l}\text{Quasi-separativity}\\ \text{Weakly balancity}\end{array}} \arrow[r, Rightarrow] & \boxed{\text{Quasi-separativity}} \\
\boxed{\text{Cancellativity}} \arrow[r, Rightarrow] \arrow[u, Rightarrow] & \boxed{\text{Weak cancellativity}} \arrow[r, Rightarrow] \arrow[u, Rightarrow] & \boxed{\begin{array}{l}\text{Quasi-separativity}\\ \text{Quasi-cancellativity}\end{array}} \arrow[u, Rightarrow]
\end{tikzcd}$$

Now we shall show that all implications in this picture are strict.

Obviously, any commutative quasi-cancellative semigroup is cancellative. Hence not every separative semigroup is quasi-cancellative. From here it follows that all the vertical implications are strict.

Every completely simple semigroup is weakly cancellative, but not separative (if it is not a group). Hence the left horizontal implications are strict.

Bicyclic semigroup $B = \langle a, b \mid ba = 1 \rangle$ is quasi-separative. Since B is simple, it cannot be decomposed into a nontrivial semilattice of its subsemigroups. By Theorem 3 it is quasi-cancellative. On the other hand, the equalities

$$b^2 \cdot 1 = b^2 \cdot ab, \qquad 1 \cdot a = ab \cdot a$$

imply that B is not weakly balanced. From here it follows that the right horizontal implications are strict.

References

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Lugansk National Pedagogical University

kji@leasat.net

Kharkov National University

boris.v.novikov@univer.kharkov.ua